

# $\theta_{13}$ , $\mu\tau$ symmetry breaking and neutrino Yukawa textures

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Within the type-I see-saw and in the basis where charged lepton and heavy neutrino mass matrices are real and diagonal,  $\mu\tau$  symmetric four and three zero neutrino Yukawa textures are perturbed by the most general  $\mu\tau$  symmetry breaking terms treated to the lowest order. For reasonable values of those symmetry breaking parameters, current best-fit ranges of neutrino mass squared differences and mixing angles are shown to be accommodable, including a  $\theta_{13} \sim 9^\circ$ , provided the light neutrinos have an inverted mass ordering.

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A major recent development in Particle Physics has been the observation [1] of a nonzero mixing between the first and third generations of (anti-) neutrinos with a measured angle  $\theta_{13} \sim 9^\circ$ . The underlying physical implication is rather serious. Certain flavor symmetries in the neutrino sector, such as [2] that under  $\mu \leftrightarrow \tau$  interchange [3], must be broken. The latter became a highly popular idea on account of its prediction of a maximal mixing ( $\theta_{23} = 45^\circ$ ) between the second and third generations of neutrinos — a situation still well-allowed by extant data. But it also predicted a vanishing  $\theta_{13}$  which has now been experimentally excluded. Hence arises the interest in breaking this  $\mu\tau$  symmetry. Since its spontaneous breakdown would require several additional fields [4], a minimalist approach would be to try explicit breaking. Instead of introducing [5] such symmetry breaking parameters directly into the derived neutrino mass matrix  $M_\nu$ , we prefer to do so in the neutrino Yukawa coupling matrix which appears in the Lagrangian itself. The latter is the more basic way of handling any explicit symmetry breaking.

In this note we treat the effect of such an explicit breaking of  $\mu\tau$  symmetry on allowed and predictive neutrino Yukawa textures. The latter are configurations with zeros of the neutrino Yukawa coupling matrix or equivalently of the neutrino Dirac mass matrix  $M_D$ . These provide a useful and effective framework within which to discuss neutrino mass and mixing parameters. Given the distinguished record [6] of presumed four zero Yukawa textures in the quark sector, it is natural to extend similar ideas to neutrinos. But one needs to take into account the difference here due to the type-I see-saw mechanism [7] which we assume to explain the observed smallness of neutrino masses. The relevant point now is that, unlike for those put [5] directly into  $M_\nu$ , *small* values of  $\mu\tau$  symmetry breaking parameters introduced into  $M_D$  can realistically generate  $\theta_{13} \sim 9^\circ$  for predictive neutrino Yukawa textures [8]. But we find the remarkable result that this is possible *only with an inverted mass ordering* of the light neutrinos.

The mass terms in our starting Lagrangian are

$$-\mathcal{L}^m = \frac{1}{2} \begin{pmatrix} \bar{\nu}_L^0 & \overline{N^0}^C_L \end{pmatrix} \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} \nu_R^{0C} \\ N_R^0 \end{pmatrix} + \bar{l}_L^0 M_l l_R^0 + h.c., \quad (1)$$

In (1) the superscript “0” signifies a flavor eigenstate field,  $\nu_L$  ( $N_R$ ) stands for three left chiral (right chiral) neutrinos while  $l$  denotes the three charged leptons. Each  $M$  is a  $3 \times 3$  matrix in flavor space, but the eigenvalues  $M_1, M_2, M_3$  of the Majorana mass-matrix  $M_R$  are much bigger ( $> 10^9$  GeV) than those of the Dirac mass matrices  $M_D$  and  $M_l$ . We work, without loss of generality, in the weak basis defined by real and diagonal  $M_R$  and  $M_l$ :

$$M_R = \text{diag}(M_1, M_2, M_3), \\ M_l = \text{diag}(m_e, m_\mu, m_\tau). \quad (2)$$

The see-saw formula for the ultralight neutrino Majoran mass matrix  $M_\nu$  is

$$M_\nu \simeq -M_D M_R^{-1} M_D^T. \quad (3)$$

Furthermore,

$$U^\dagger M_\nu U^* = D \equiv \text{diag}(m_1, m_2, m_3), \quad (4)$$

where  $m_1, m_2, m_3$  are the light neutrino masses and we follow the PDG [9] convention in defining them. Then, for

$$H_\nu = M_\nu M_\nu^\dagger, \\ U^\dagger H_\nu U = D^2 \quad (5)$$

with  $U$  being the unitary PMNS matrix.

Because of  $\mu\tau$  symmetry,  $M_D$  and  $M_R$  remain invariant under the interchange of  $\mu$  (2) and  $\tau$  (3) flavors. Thus, in our basis,  $M_D$  and  $M_R$  have the general forms

$$M_D = \begin{pmatrix} a & b & b \\ c & d & e \\ c & e & d \end{pmatrix},$$

$$M_R = \text{diag} (M_1, M_2, M_2). \quad (6)$$

Though  $M_{1,2}$  are real,  $a, b, c, d, e$  in (6) are five complex parameters. We now set up two additional criteria. First we take every ultralight neutrino to have a nonzero mass, i.e.

$$\det M_D \neq 0. \quad (7)$$

Second, we utilize the observed fact that none of the neutrino families decouples from any of the other two. Given these features, four has been shown [10] to be the maximum number of zeros that  $M_D$  can accommodate. These four zero textures are phenomenologically quite interesting [11] and also can effect desired baryogenesis via leptogenesis at a high scale [12]. We discuss here the phenomenological consequences of four and three zero textures of  $M_D$  in a general explicitly broken (to lowest order)  $\mu\tau$  symmetric set-up. We do not consider textures with a lower number of zeros since they contain too many parameters and have little predictivity. Apart from the '11' element 'a' in  $M_D$ , cf.(6), the other complex parameters come in pairs. So, for an even (odd) number of zeros of  $M_D$ , 'a' must be nonvanishing (vanishing).

We consider the four zero and three zero cases separately.

(a) **Four zero textures** can come in  ${}^4C_2 = 6$  ways. Of these, two are ruled out by the twin criteria set up in the earlier paragraph. The four allowed  $\mu\tau$  symmetric four zero textures fall into two categories  $A$  and  $B$ , each having one  $M_\nu$  [13], and can be written, after appropriate relabelling, as

$$M_{DA1}^{(4)} = \begin{pmatrix} a & b & b \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix}, \quad M_{DA2}^{(4)} = \begin{pmatrix} a & b & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix},$$

$$M_{DB1}^{(4)} = \begin{pmatrix} a & 0 & 0 \\ b & 0 & c \\ b & c & 0 \end{pmatrix}, \quad M_{DB2}^{(4)} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ b & 0 & c \end{pmatrix}. \quad (8)$$

(b) **Three zero textures**, with a necessarily vanishing element 'a', have only two surviving textures once the twin criteria are imposed. Either has the same  $M_\nu$ . The textures are

$$M_{DC1}^{(3)} = \begin{pmatrix} 0 & b & b \\ c & 0 & d \\ c & d & 0 \end{pmatrix}$$

$$M_{DC2}^{(3)} = \begin{pmatrix} 0 & b & b \\ c & d & 0 \\ c & 0 & d \end{pmatrix} \quad (9)$$

and we put them under category  $C$ .

The superscripts in the left hand sides (8) and (9) refer to the number of zeros presesnt. It is straightforward to write in a simplified way the corresponding neutrino mass matrices. Introducing two real variables and one phase for each category (vide Table I), i.e.  $(k_1, k_2, \alpha)$  for  $M_{\nu A}^{(4)}$ ,  $(l_1, l_2, \beta)$  for  $M_{\nu B}^{(4)}$  and  $(r_1, r_2, \gamma)$  for  $M_{\nu C}^{(3)}$  plus  $m_A, m_B$  and  $m_C$  as overall scales for the three cases, we have

$$M_{\nu A}^{(4)} = m_A \begin{pmatrix} k_1^2 e^{2i\alpha} + 2k_2^2 & k_2 & k_2 \\ k_2 & 1 & 0 \\ k_2 & 0 & 1 \end{pmatrix},$$

$$M_{\nu B}^{(4)} = m_B \begin{pmatrix} l_1^2 & l_1 l_2 e^{i\beta} & l_1 l_2 e^{i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} + 1 & l_2^2 e^{2i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} & l_2^2 e^{2i\beta} + 1 \end{pmatrix}$$

$$M_{\nu C}^{(3)} = m_C \begin{pmatrix} 2r_1^2 & r_1 & r_1 \\ r_1 & r_2^2 e^{2i\gamma} + 1 & r_2^2 e^{2i\gamma} \\ r_1 & r_2^2 e^{2i\gamma} & r_2^2 e^{2i\gamma} + 1 \end{pmatrix}. \quad (10)$$

Note that in writing (10), certain phases,  $\alpha'$  for  $M_{\nu A}^{(4)}$  (cf. Table 1),  $\beta'$  for  $M_{\nu B}^{(4)}$  and  $\gamma'$  for  $M_{\nu C}^{(3)}$ , have been rotated away by redefining the  $\nu_e$  field. We have also introduced in Table I certain functions  $X_1$ ,  $X_2$  and  $X_3$  of the said variables

TABLE I: Definitions of differenet parameters involved in  $M_\nu$  for all cases in (10).

Four zero textures		Three zero textures
Category A	Category B	Category C
$m_A = -c^2/M_2,$ $k_1 e^{i(\alpha+\alpha')} = \frac{a}{c} \sqrt{\frac{M_2}{M_1}},$ $k_2 e^{i\alpha'} = \frac{b}{c},$ $\alpha = \arg \frac{a}{b}$	$m_B = -c^2/M_2,$ $l_1 e^{i\beta'} = \frac{a}{c} \sqrt{\frac{M_2}{M_1}}$ $l_2 e^{i\beta} = \frac{b}{c} \sqrt{\frac{M_2}{M_1}}$ $\beta = \arg \frac{b}{c}$	$m_C = -d^2/M_2$ $r_1 e^{i\gamma'} = \frac{b}{d}$ $r_2 e^{i\gamma} = \frac{c}{d} \sqrt{\frac{M_2}{M_1}}$ $\gamma = \arg \frac{c}{d}$
$X_1 = 2\sqrt{2}k_2[(1+2k_2^2)^2 + k_1^4 + 2k_1^2(1+2k_2^2)\cos 2\alpha]^{1/2}$ $X_2 = 1 - k_1^4 - 4k_2^4 - 4k_1^2k_2^2\cos 2\alpha$ $X_3 = 1 - 4k_2^4 - k_1^4 - 4k_1^2k_2^2\cos 2\alpha - 4k_2^2$	$X_1 = 2\sqrt{2}l_1l_2[(l_1^2+2l_2^2)^2 + 1 + 2(l_1^2+2l_2^2)\cos 2\beta]^{1/2}$ $X_2 = 1 + 4l_2^2\cos 2\beta + 4l_2^4 - l_1^4$ $X_3 = 1 - (l_1^2+2l_2^2)^2 - 4l_2^2\cos 2\beta$	$X_1 = 2\sqrt{2}r_1[(1+2r_1^2)^2 + 4r_2^4 + 4r_2^2(1+2r_1^2)\cos 2\gamma]^{1/2}$ $X_2 = 4r_2^4 + 1 + 4r_2^2\cos 2\gamma - 4r_1^4$ $X_3 = 1 - 4r_1^4 - 4r_1^2 - 4r_2^4 - 4r_2^2\cos 2\gamma$

for each category. These relate to measurable quantities by rather simple formulae. If we define  $X = \sqrt{X_1^2 + X_2^2}$ , then

$$\Delta_{21}^2 = m^2 X, \quad (11a)$$

$$\Delta_{32}^2 = \frac{m^2}{2}(X_3 - X), \quad (11b)$$

$$\tan 2\theta_{12} = \frac{X_1}{X_2}, \quad (11c)$$

$$m_{1,2} = \left| \Delta_{21}^2 \left( \frac{2 - X_3 \mp X}{2X} \right) \right|^{1/2}, \quad (11d)$$

$$m_3 = |\Delta_{21}^2/X|^{1/2}. \quad (11e)$$

In (11)  $m$  equals  $m_A$ ,  $m_B$  or  $m_C$ , as defined in Table I, depending on the category.

We now introduce explicit  $\mu\tau$  symmetry breaking into  $M_D$  and  $M_R$  as follows. A real parameter  $\delta$  is first inserted in  $M_R$  of (6) which gets modified to

$$M_R^\delta = \text{diag} [M_1, M_2, M_2(1-\delta)]. \quad (12)$$

Additional real parameters  $\epsilon_i$  and corresponding phases  $\phi_i$  are introduced into the textures describing  $M_D^\epsilon$  with with  $(\epsilon_1, \phi_1)$ ,  $(\epsilon_2, \phi_2)$  and  $(\epsilon_3, \phi_3)$  modifying elements  $b, c$  and  $d$  respectively in the first two rows of  $M_D$  only. Consequently, we have

$$\begin{aligned}
M_{DA1}^{\epsilon(4)} &= \begin{pmatrix} a & b & b(1-\epsilon_1 e^{i\phi_1}) \\ 0 & 0 & c(1-\epsilon_2 e^{i\phi_2}) \\ 0 & c & 0 \end{pmatrix}, & M_{DA2}^{\epsilon(4)} &= \begin{pmatrix} a & b & b(1-\epsilon_1 e^{i\phi_1}) \\ 0 & c(1-\epsilon_2 e^{i\phi_2}) & 0 \\ 0 & 0 & c \end{pmatrix}, \\
M_{DB1}^{\epsilon(4)} &= \begin{pmatrix} a & 0 & 0 \\ b(1-\epsilon_1 e^{i\phi_1}) & 0 & c(1-\epsilon_2 e^{i\phi_2}) \\ b & c & 0 \end{pmatrix}, & M_{DB2}^{\epsilon(4)} &= \begin{pmatrix} a & 0 & 0 \\ b(1-\epsilon_1 e^{i\phi_1}) & c(1-\epsilon_2 e^{i\phi_2}) & 0 \\ b & 0 & c \end{pmatrix}, \\
M_{DC1}^{\epsilon(3)} &= \begin{pmatrix} 0 & b & b(1-\epsilon_1 e^{i\phi_1}) \\ c(1-\epsilon_2 e^{i\phi_2}) & 0 & d(1-\epsilon_3 e^{i\phi_3}) \\ c & d & 0 \end{pmatrix}, & M_{DC2}^{\epsilon(3)} &= \begin{pmatrix} 0 & b & b(1-\epsilon_1 e^{i\phi_1}) \\ c(1-\epsilon_2 e^{i\phi_2}) & d(1-\epsilon_3 e^{i\phi_3}) & 0 \\ c & 0 & d \end{pmatrix}. \quad (13)
\end{aligned}$$

On account of (3), the neutrino mass matrix  $M_\nu$  develops the general broken  $\mu\tau$  symmetric form

$$M_\nu^{\epsilon,\delta} = m \left[ \begin{pmatrix} P & Q & Q \\ Q & R & S \\ Q & R & S \end{pmatrix} - \epsilon_1 e^{i\phi_1} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} - \epsilon_2 e^{i\phi_2} \begin{pmatrix} y_1 & y_2 & y_3 \\ y_2 & y_4 & y_5 \\ y_3 & y_5 & y_6 \end{pmatrix} - \epsilon_3 e^{i\phi_3} \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} - \delta \begin{pmatrix} t_1 & t_2 & t_3 \\ t_2 & t_4 & t_5 \\ t_3 & t_5 & t_6 \end{pmatrix} \right]. \quad (14)$$

There are now six  $M_\nu^{\epsilon,\delta}$ 's in total following from the six  $M_D$ 's of (13). In (14),  $m$  is as in Table I while  $\epsilon_i$ ,  $\phi_i$  and  $\delta$  are universal  $\mu\tau$  symmetry breaking parameters, introduced in (12) and (13) with  $\epsilon_3$  evidently vanishing for four zero textures. In contrast, the (generally complex) quantities  $P$ ,  $Q$ ,  $R$ ,  $S$  as well as  $x_{1,...,6}$ ,  $y_{1,...,6}$ ,  $z_{1,...,6}$ ,  $t_{1,...,6}$  vary from category to category and are listed in Table II in terms of the real parameters  $k_{1,2}$ ,  $l_{1,2}$ ,  $r_{1,2}$  and phases  $\alpha$ ,  $\beta$ ,  $\gamma$  with broken  $\mu\tau$  symmetry. There are four distinct  $M_\nu$ 's for four zero and two distinct ones for three zero textures. In the limit of vanishing  $\delta$  and  $\epsilon_{1,2,3}$ , the  $\mu\tau$  symmetric form of  $M_\nu$ 's in terms  $P$ ,  $Q$ ,  $R$ ,  $S$  is restored and we recover two  $M_\nu$ 's ( a single  $M_\nu$ ) for four (three) zero textures. Next, we direct our attention to experimentally measurable

TABLE II: Expressions for quantities appearing in  $M_\nu^{\epsilon,\delta}$ . Parameters  $z_{1-6}$ , not needed for four zero textures since  $\epsilon_3 = 0$ , have been kept blank for the latter.

Quanatity	Four zero				Three zero	
	Category A1	Category A2	Category B1	Category B2	Case 1	Case 2
$P$	$k_1^2 e^{2i\alpha} + 2k_2^2$	$k_1^2 e^{2i\alpha} + 2k_2^2$	$l_1^2$	$l_1^2$	$2r_1^2$	$2r_1^2$
$Q$	$k_2$	$k_2$	$l_1 l_2 e^{i\beta}$	$l_1 l_2 e^{i\beta}$	$r_1$	$r_1$
$R$	1	1	$l_2^2 e^{2i\beta} + 1$	$l_2^2 e^{2i\beta} + 1$	$r_2^2 e^{2i\gamma} + 1$	$r_2^2 e^{2i\gamma} + 1$
$S$	0	0	$l_2^2 e^{2i\beta}$	$l_2^2 e^{2i\beta}$	$r_2^2 e^{2i\gamma}$	$r_2^2 e^{2i\gamma}$
$x_1$	$2k_2^2$	$2k_2^2$	0	0	$2r_1^2$	$2r_1^2$
$x_2$	$k_2$	0	$l_1 l_2 e^{i\beta}$	$l_1 l_2 e^{i\beta}$	0	$r_1$
$x_3$	0	$k_2$	0	0	$r_1$	0
$x_4$	0	0	$2l_2^2 e^{2i\beta}$	$2l_2^2 e^{2i\beta}$	0	0
$x_5$	0	0	$l_2^2 e^{2i\beta}$	$l_2^2 e^{2i\beta}$	0	0
$x_6$	0	0	0	0	0	0
$y_1$	0	0	0	0	0	0
$y_2$	$k_2$	$k_2$	0	0	0	0
$y_3$	0	0	0	0	0	0
$y_4$	2	2	2	2	$2r_2^2 e^{2i\gamma}$	$2r_2^2 e^{2i\gamma}$
$y_5$	0	0	0	0	$r_2^2 e^{2i\gamma}$	$r_2^2 e^{2i\gamma}$
$y_6$	0	0	0	0	0	0
$z_1$	-	-	-	-	0	0
$z_2$	-	-	-	-	$r_1$	$r_1$
$z_3$	-	-	-	-	0	0
$z_4$	-	-	-	-	2	2
$z_5$	-	-	-	-	0	0
$z_6$	-	-	-	-	0	0
$t_1$	$-k_2^2$	$-k_2^2$	0	0	$-r_1^2$	$-r_1^2$
$t_2$	$-k_2$	0	0	0	0	$-r_1$
$t_3$	0	$-k_2$	0	0	$-r_1$	0
$t_4$	-1	0	-1	0	0	-1
$t_5$	0	0	0	0	0	0
$t_6$	0	-1	0	-1	-1	0

quantities which are best related to elements of the matrix  $H_\nu$  of (5). From (13)

$$\begin{aligned}
H_\nu^{\epsilon,\delta} = & m^2 \left[ \begin{pmatrix} |P|^2 + 2|Q|^2 & PQ^* + Q(R^* + S^*) & PQ^* + Q(R^* + S^*) \\ P^*Q + Q^*(R + S) & |Q|^2 + |R|^2 + |S|^2 & |Q|^2 + RS^* + R^*S \\ P^*Q + Q^*(R + S) & |Q|^2 + R^*S + RS^* & |Q|^2 + |R|^2 + |S|^2 \end{pmatrix} \right. \\
& \left. - \epsilon_1 \begin{pmatrix} u_1 & u_2^* & u_3^* \\ u_2 & u_4 & u_5^* \\ u_3 & u_5 & u_6 \end{pmatrix} - \epsilon_2 \begin{pmatrix} v_1 & v_2^* & v_3^* \\ v_2 & v_4 & v_5^* \\ v_3 & v_5 & v_6 \end{pmatrix} - \epsilon_3 \begin{pmatrix} w_1 & w_2^* & w_3^* \\ w_2 & w_4 & w_5^* \\ w_3 & w_5 & w_6 \end{pmatrix} - \delta \begin{pmatrix} s_1 & s_2^* & s_3^* \\ s_2 & s_4 & s_5^* \\ s_3 & s_5 & s_6 \end{pmatrix} \right].
\end{aligned} \tag{15}$$

We have introduced in (15) the quantities  $u_i$ ,  $v_i$ ,  $w_i$  and  $s_i$  ( $i = 1, \dots, 6$ ) which are algebraic functions of the variables of (14), i.e.  $P$ ,  $Q$ ,  $R$ ,  $S$ ,  $x_i$ ,  $y_i$ ,  $z_i$ ,  $t_i$ ,  $\phi_1$ ,  $\phi_2$ ,  $\phi_3$ . Evidently,  $u_k$ ,  $v_k$ ,  $w_k$ ,  $s_k$  ( $k = 1, 4, 6$ ) are real and the rest are complex.

The first point to note is this. With  $i$  and  $j$  running from 1 to 6,  $u_i$  do not involve  $y_j$ ,  $z_j$ ,  $t_j$ ,  $\phi_2$ ,  $\phi_3$ ; similarly,  $v_i$  do not involve  $x_j$ ,  $z_j$ ,  $t_j$ ,  $\phi_3$ ,  $\phi_1$ ;  $w_i$  do not involve  $x_j$ ,  $y_j$ ,  $t_j$ ,  $\phi_1$ ,  $\phi_2$ ;  $s_i$  do not involve  $x_j$ ,  $y_j$ ,  $z_j$ ,  $\phi_1$ ,  $\phi_2$ ,  $\phi_3$ . Furthermore,

$u_i$ ,  $v_i$ ,  $w_i$  and  $s_i$  are related by certain substitution relations. With  $u_i$  written as a set of functions  $f_i$  of certain variables,  $v_i$ ,  $w_i$  and  $s_i$  are the same functions but of a substituted set of variables. Thus

$$\begin{aligned} u_i &= f_i(x_1, x_2, \dots, x_6, \phi_1), \\ v_i &= f_i(y_1, y_2, \dots, y_6, \phi_2), \\ w_i &= f_i(z_1, z_2, \dots, z_6, \phi_3), \\ s_i &= f_i(t_1, t_2, \dots, t_6, 0). \end{aligned} \quad (16)$$

It is thus sufficient to explicitly display the RHS of the first equation in (16):

$$\begin{aligned} u_1 &= [P^*x_1 + Q^*(x_2 + x_3)]e^{i\phi_1} + c.c., \\ u_2 &= [P^*x_2 + Q^*(x_4 + x_5)]e^{i\phi_1} + [Qx_1^* + Rx_2^* + Sx_3^*]e^{-i\phi_1}, \\ u_3 &= [P^*x_3 + Q^*(x_5 + x_6)]e^{i\phi_1} + [Qx_1^* + Sx_2^* + Rx_3^*]e^{-i\phi_1}, \\ u_4 &= [Q^*x_2 + R^*x_4 + S^*x_5]e^{i\phi_1} + c.c., \\ u_5 &= [Q^*x_3 + R^*x_5 + S^*x_6]e^{i\phi_1} + [Qx_2^* + Sx_4^* + Rx_5^*]e^{-i\phi_1}, \\ u_6 &= [Q^*x_3 + S^*x_5 + R^*x_6]e^{i\phi_1} + c.c. \end{aligned} \quad (17)$$

The diagonalization of  $H_\nu^{\epsilon, \delta}$ , as given (15), yields not only the squares of the physical neutrino masses after  $\mu\tau$  symmetry breaking, but also the mixing angles. We wish to compute  $(\Delta_{21}^2)^{\epsilon, \delta} \equiv (m_2^{\epsilon, \delta})^2 - (m_1^{\epsilon, \delta})^2$ ,  $(\Delta_{32}^2)^{\epsilon, \delta} \equiv (m_3^{\epsilon, \delta})^2 - (m_2^{\epsilon, \delta})^2$ ,  $\theta_{12}^{\epsilon, \delta}$ ,  $\theta_{23}^{\epsilon, \delta}$ ,  $\theta_{13}^{\epsilon, \delta}$ . The superscripts here signify that the concerned physical quantities have been calculated with  $\mu\tau$  symmetry breaking taken into account. The complicated algebraic expressions for these quantities can be simplified by defining another set of functions  $U_i$ ,  $V_i$ ,  $W_i$ ,  $S_i$  ( $i = 1, \dots, 6$ ) of the quantities introduced in (16). Once again, each of these depends only on a subset of the set  $\{u_i, v_i, w_i, s_i\}$  and is related to the other three by substitution rules. Thus,

$$\begin{aligned} U_i &= F_i(u_1, u_2, \dots, u_6), \\ V_i &= F_i(v_1, v_2, \dots, v_6), \\ W_i &= F_i(w_1, w_2, \dots, w_6), \\ S_i &= F_i(s_1, s_2, \dots, s_6). \end{aligned} \quad (18)$$

with the same functional form  $F_i$ . Again, it suffices just to explicitly specify the form of  $U_i$ :

$$\begin{aligned} U_1 &= \frac{1}{2} \left[ -2c_{12}^2 u_1 + \sqrt{2}c_{12}s_{12} \{ (u_2 + u_3)e^{-i\psi} + (u_2^* + u_3^*)e^{i\psi} \} - s_{12}^2(u_4 + u_6 + u_5 + u_5^*) \right], \\ U_2 &= \frac{1}{2} \left[ -\sqrt{2}c_{12}^2(u_2 + u_3)e^{-i\psi} + \sqrt{2}s_{12}^2(u_2^* + u_3^*)e^{i\psi} + c_{12}s_{12}(u_4 + u_6 - 2u_1 + u_5 + u_5^*) \right], \\ U_3 &= \frac{1}{2} \left[ \sqrt{2}c_{12}(u_2 - u_3)e^{-i\psi} + s_{12}(u_6 - u_4 + u_5 - u_5^*) \right], \\ U_4 &= \frac{1}{2} \left[ -2s_{12}^2 u_1 - \sqrt{2}c_{12}s_{12} \{ (u_2 + u_3)e^{-i\psi} + (u_2^* + u_3^*)e^{i\psi} \} - c_{12}^2(u_4 + u_6 + u_5 + u_5^*) \right], \\ U_5 &= \frac{1}{2} \left[ \sqrt{2}s_{12}(u_2 - u_3)e^{-i\psi} - c_{12}(u_6 - u_4 + u_5 - u_5^*) \right], \\ U_6 &= \frac{1}{2} [u_5 + u_5^* - u_4 - u_6]. \end{aligned} \quad (19)$$

In (19)  $c_{12} = \cos \theta_{12}$ ,  $s_{12} = \sin \theta_{12}$ , with  $\theta_{12}$  as the unperturbed mixing angle between the first two generations, as given in (11). Moreover, the phase  $\psi$  is given by

$$\psi = \arg [P^*Q + Q^*(R + S)]. \quad (20)$$

Finally, we can display the measured quantities in terms of their unperturbed values:

$$(m_1^{\epsilon,\delta})^2 = m_1^2 + m^2 [U_1\epsilon_1 + V_1\epsilon_2 + W_1\epsilon_3 + S_1\delta], \quad (21a)$$

$$(m_2^{\epsilon,\delta})^2 = m_2^2 + m^2 [U_4\epsilon_1 + V_4\epsilon_2 + W_4\epsilon_3 + S_4\delta], \quad (21b)$$

$$(m_3^{\epsilon,\delta})^2 = m_3^2 + m^2 [U_6\epsilon_1 + V_6\epsilon_2 + W_6\epsilon_3 + S_6\delta]. \quad (21c)$$

$$(\Delta_{21}^2)^{\epsilon,\delta} = \Delta_{21}^2 + m^2 \{(U_4 - U_1)\epsilon_1 + (V_4 - V_1)\epsilon_2 + (W_4 - W_1)\epsilon_3 + (S_4 - S_1)\delta\}, \quad (21d)$$

$$(\Delta_{32}^2)^{\epsilon,\delta} = \Delta_{32}^2 + m^2 \{(S_6 - S_4)\delta + (U_6 - U_4)\epsilon_1 + (V_6 - V_4)\epsilon_2 + (W_6 - W_4)\epsilon_3\}, \quad (21e)$$

$$(\sin \theta_{12})^{\epsilon,\delta} = \left| s_{12} + c_{12}m^2 \left\{ \frac{S_2^*\delta + U_2^*\epsilon_1 + V_2^*\epsilon_2 + W_2^*\epsilon_3}{\Delta_{21}^2} \right\} \right|, \quad (21f)$$

$$(\sin \theta_{23})^{\epsilon,\delta} = \left| \frac{1}{\sqrt{2}} + \frac{s_{12}m^2}{\sqrt{2}} \left\{ \frac{S_3^*\delta + U_3^*\epsilon_1 + V_3^*\epsilon_2 + W_3^*\epsilon_3}{\Delta_{21}^2 + \Delta_{32}^2} \right\} - \frac{c_{12}m^2}{\sqrt{2}} \left\{ \frac{S_5^*\delta + U_5^*\epsilon_1 + V_5^*\epsilon_2 + W_5^*\epsilon_3}{\Delta_{32}^2} \right\} \right|, \quad (21g)$$

$$(\sin \theta_{13})^{\epsilon,\delta} = \left| c_{12}m^2 \left\{ \frac{S_3^*\delta + U_3^*\epsilon_1 + V_3^*\epsilon_2 + W_3^*\epsilon_3}{\Delta_{21}^2 + \Delta_{32}^2} \right\} + s_{12}m^2 \left\{ \frac{S_5^*\delta + U_5^*\epsilon_1 + V_5^*\epsilon_2 + W_5^*\epsilon_3}{\Delta_{32}^2} \right\} \right|. \quad (21h)$$

$$(21i)$$

We can also write an equation for the CP-violating Jarlskog invariant which is nonzero only because of  $\mu\tau$  symmetry breaking:

$$J_{\text{CP}} = \text{Im} \frac{(H_\nu^{\epsilon,\delta})_{12}(H_\nu^{\epsilon,\delta})_{23}(H_\nu^{\epsilon,\delta})_{31}}{(\Delta_{21}^2)^{\epsilon,\delta}(\Delta_{32}^2)^{\epsilon,\delta}(\Delta_{31}^2)^{\epsilon,\delta}} \quad (22)$$

$$= \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta_D. \quad (23)$$

where  $\delta_D$  is the Dirac phase.

#### Numerical results and discussion

The breaking of  $\mu\tau$  symmetry generates a nonzero  $\theta_{13}$  as well as makes  $\theta_{23}$  deviate from its maximal value. In the present work, we vary both the mass squared differences and the three mixing angles within their  $3\sigma$  experimental ranges. Thus we treat both  $\theta_{13}$  and  $\theta_{23}$  as inputs, remaining constrained within given intervals. On feeding in the experimental  $3\sigma$  ranges of the quantities listed in the Table III, one can check which of the six  $M_\nu^{\epsilon,\delta}$ 's can accommodate them. Let us first discuss the four zero textures of categories  $A_1$ ,  $A_2$ ,  $B_1$  and  $B_2$ . We find that just the one  $M_\nu^{\epsilon,\delta}$  originating from  $M_{DA1}^{\epsilon(4)}$  survives, *the other three do not*. This texture, which involves the real parameters  $k_{1,2}$  and the phase  $\alpha$ , allows only an inverted ordering of the light neutrino masses:  $\Delta_{32}^2 < 0$ . The phase  $\alpha$  is found to be restricted to  $89^\circ < \alpha < 90^\circ$ . The  $\mu\tau$  symmetry breaking real parameters are confined to  $0.08 \lesssim \epsilon_1 \lesssim 0.15$ ,  $0 \lesssim \epsilon_2 \lesssim 0.1$ ,  $0 \lesssim \delta \lesssim 0.1$ . The nonzero lower bound on  $\epsilon_1$  is controlled most sensitively by the allowed lower bound on  $\theta_{13}$ . We find  $\epsilon_1$  to be the main parameter which drives the breaking of  $\mu\tau$  symmetry and have restricted its maximum value to 0.15. Now the upper bounds on the other two parameters  $\epsilon_2$  and  $\delta$  stem from the requirement of the observed fact  $m_2 > m_1$  and from the allowed range of  $\theta_{12}$ . Regarding the corresponding phases  $\phi_{1,2}$ ,  $\phi_2$  is completely unrestricted while the allowed range of  $\phi_1$  is  $85^\circ \lesssim \phi_1 \lesssim 95^\circ$ . We follow an identical procedure with the two allowed three zero textures. Only the  $M_\nu^{\epsilon,\delta}$ , originating from  $M_{DC1}^{\epsilon(3)}$ , i.e. category  $C1$ , is found to survive with an inverted light neutrino mass ordering  $\Delta_{32}^2 < 0$ ; the other one, namely  $C2$ , *does not*. Here also the phase  $\gamma$  is found to be restricted to  $89^\circ < \gamma < 90^\circ$ . The  $\mu\tau$  symmetry breaking real parameters are confined to  $0 \lesssim \epsilon_1 \lesssim 0.15$ ,  $0 \lesssim \epsilon_2 \lesssim 0.02$ ,  $0.05 \lesssim \epsilon_3 \lesssim 0.1$ ,  $0.06 \lesssim \delta \lesssim 0.1$ .

TABLE III: Input experimental values [15]

Quantity	Experimental $3\sigma$ range
$\Delta_{21}^2$	$7.12 \times 10^{-5} \text{ eV}^2 < \Delta_{21}^2 < 8.20 \times 10^{-5} \text{ eV}^2$
$\Delta_{32}^2 < 0$	$-2.72 \times 10^{-3} \text{ eV}^2 < \Delta_{32}^2 < -2.28 \times 10^{-3} \text{ eV}^2$
$\Delta_{32}^2 > 0$	$2.23 \times 10^{-3} \text{ eV}^2 < \Delta_{32}^2 < 2.67 \times 10^{-3} \text{ eV}^2$
$\theta_{12}$	$31.30^\circ < \theta_{12} < 37.46^\circ$
$\theta_{23}$	$36.86^\circ < \theta_{23} < 55.55^\circ$
$\theta_{13}$	$7.49^\circ < \theta_{13} < 10.46^\circ$
$\delta_D$	Unknown

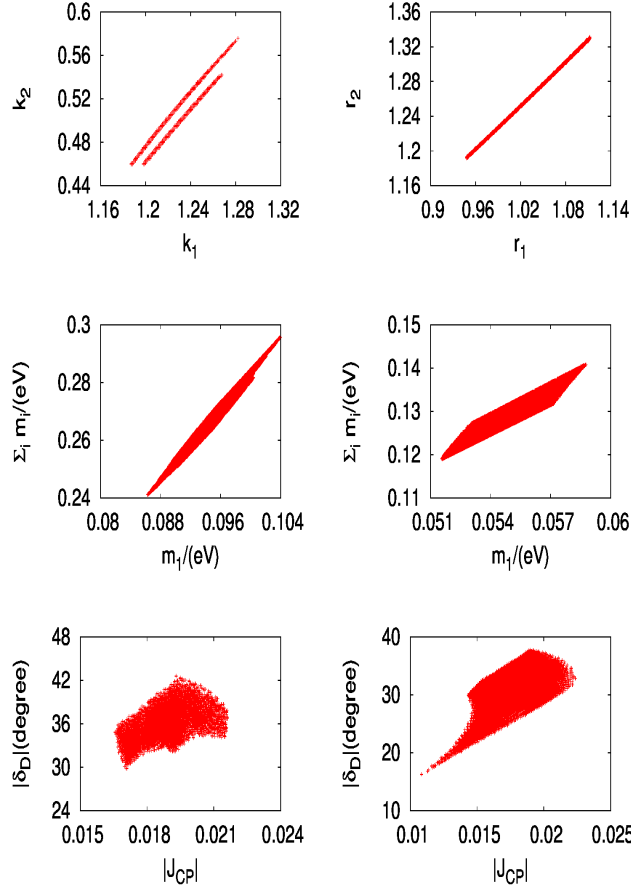


FIG. 1: (Color online) Allowed real parameters(top),  $\sum_i m_i$  vs  $m_1$  (middle) and  $|J_{CP}|$  vs  $|\delta_D|$  (bottom) for category A1 (left) of four zero and category C1 (right) of three zero textures.

The lower bounds on  $\epsilon_3$ ,  $\delta$  are controlled by the lower bound on  $\theta_{13}$ . As before, the upper bound on  $\epsilon_1$  has been fixed by hand while those on  $\epsilon_2$ ,  $\epsilon_3$  and  $\delta$  are controlled by the condition  $m_2 > m_1$  and the allowed range of  $\theta_{12}$ . The phases  $\phi_1$  and  $\phi_2$  are unrestricted but  $\phi_3$  turns out to be confined within  $0 < \phi_3 < \pi/4$ .

We provide some sample plots in Fig. 1 for specific allowed values of  $\epsilon_i$ ,  $\phi_i$  and  $\delta$ . For the four zero texture of category A1, we have chosen  $\epsilon_1 = 0.15$ ,  $\epsilon_2 = 0.0$ ,  $\delta = 0.11$ ,  $\phi_1 = \phi_2 = \pi/4$  and for the three zero texture of category C1, the values are  $\epsilon_1 = \epsilon_2 = 0.0$ ,  $\epsilon_3 = 0.08$ ,  $\delta = 0.08$ ,  $\phi_1 = \pi/2$ ,  $\phi_2 = \pi/4$ ,  $\phi_3 = 0$ . The top left panel of Fig. 1 displays the allowed values in the  $k_1 - k_2$  plane for category A1. Note that only rather small positive values of these real parameters are allowed. The middle left panel shows the sum of the light neutrino masses plotted against  $m_1$  and the bottom panel shows the modulus of the Jarlskog invariant  $J_{CP}$ , a purely  $\mu\tau$  symmetry breaking effect, vs. the absolute value of the Dirac phase  $\delta_D$ . It is interesting that the neutrino mass sum stays between 0.24 eV and 0.3 eV while  $|\delta_D|$  remains between  $30^\circ$  and  $42^\circ$  with  $|J_{CP}| \sim 2 \times 10^{-2}$ . The top right panel of Fig. 1 shows the allowed parametric region in the  $r_1 - r_2$  plane for category C1. Here the values are near unity. From the middle right panel we see that the neutrino masses are smaller in this case, their sum remaining between 0.12 eV and 0.14 eV. The ranges of  $|\delta_D|$  and  $|J_{CP}|$ , shown in the bottom right panel are rather similar to those of the other allowed category.

In conclusion, we have systematically investigated the effect of the explicit breaking of  $\mu\tau$  symmetry on allowed four and three zero neutrino Yukawa textures respecting the symmetry. We have found that, for reasonably small values of the symmetry breaking parameters, the allowed  $3\sigma$  ranges of the light neutrino mass squared differences and mixing angles can be accommodated by one four zero texture and one three zero texture, provided the neutrino mass ordering is inverted with  $\Delta_{32}^2 < 0$ . A determination that the light neutrinos are normally mass-ordered will rule out this entire scenario. On the other hand, the demonstration of an inverted light neutrino mass hierarchy will provide support.

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- [1] F. P. An *et al.* [DAYA-BAY Collaboration], Phys. Rev. Lett. **108**, 171803 (2012), J. K. Ahn *et al.* [RENO Collaboration], ibid. **108**, 191802 (2012), Y. Abe *et al.* [DOUBLE-CHOOZ Collaboration], ibid. **108**, 131801 (2012), K. Abe *et al.* [T2K Collaboration], Phys. Rev. D **85**, 031103 (2012),
- [2] For the effects of other types of discrete symmetries, see S. -F. Ge, D. A. Dicus and W. W. Repko, Phys. Rev. Lett. **108**, 041801 (2012); Phys. Lett. B **702**, 220 (2011).
- [3] For a recent review and original references therein, see W. Grimus and L. Lavoura, arXiv:1207.1678 [hep-ph].
- [4] A. S. Joshipura, B. P. Kodrani and K. M. Patel, Phys. Rev. D **79**, 115017 (2009) [arXiv:0903.2161 [hep-ph]].
- [5] A. S. Joshipura, AIP Conf. Proc. **1382**, 54 (2011).
- [6] H. Fritzsch and Z. -z. Xing, Phys. Lett. B **555**, 63 (2003), H. Fritzsch and Z. -z. Xing, Prog. Part. Nucl. Phys. **45**, 1 (2000), K. S. Babu and J. Kubo, Phys. Rev. D **71**, 056006 (2005).
- [7] P. Minkowski, Phys. Lett. B **67**, 421 (1977), M. Gell-Mann, P. Ramond and R. Slansky, in 'Supergravity', (eds. D. Friedman and P. van Nieuwenhuizen) North-Holland, Amsterdam, 1979, p.315., T. Yanagida in Proc. Workshop 'Unified Theory and Baryon Number in the Universe', eds. O. Sawada and A. Sugamoto (KEK, Tsukuba, Japan, 1979), p.95, R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980).
- [8] B. Adhikary, M. Chakraborty and A. Ghosal, Phys. Rev. D **86**, 013015 (2012)
- [9] J. Beringer *et al.* [Particle Data Group Collaboration], Phys. Rev. D **86**, 010001 (2012).
- [10] G. C. Branco, D. Emmanuel-Costa, M. N. Rebelo and P. Roy, Phys. Rev. D **77**, 053011 (2008),
- [11] S. Choubey, W. Rodejohann and P. Roy, Nucl. Phys. B **808**, 272 (2009) [Erratum-ibid. **818**, 136 (2009)].
- [12] B. Adhikary, A. Ghosal and P. Roy, JCAP **1101**, 025 (2011).
- [13] B. Adhikary, A. Ghosal and P. Roy, Mod. Phys. Lett. A **26**, 2427 (2011); JHEP **0910**, 040 (2009).
- [14] B. Brahmachari and A. Raychaudhuri, Phys. Rev. D **86**, 051302(R) (2012).
- [15] D. V. Forero, M. Tortola and J. W. F. Valle, arXiv:1205.4018 [hep-ph].